JOURNAL OF COMPUTATIONAL PHYSICS 109, 164–168 (1993)

Improvements on the Lehmer–Schur Root Detection Method

DAN LOEWENTHAL

Raymond and Beverly Sackler Faculty of Exact Sciences, Department of Geophysics, Tel Aviv University, Ramat Aviv 69978, Israel

Received July 1, 1991; revised April 8, 1993

The Schur–Cohn transformation is an important tool used to find how many roots of a polynomial are contained inside the unit circle of the complex plane. Using the same basic idea, a two-dimensional bisection scheme is derived for locating a certain root of a very high degree polynomial (typically $N > 100$). By successive applications of this transformation, the root is isolated to lie within a thin concentric annulus of width $\eta \ll 1$ centered at the origin. This procedure determines the magnitude (or modulus) of the root with accuracy η. The phase (or argument) of the root along this annulus is found by slightly perturbing the origin by an amount $\epsilon < 1$ and constructing a new concentric annulus. The intersection of the two annuli yields an estimate of the root with accuracy $2\eta/\epsilon$. The root searching scheme is global and is faster than the Lehmer–Schur direct method, since in the proposed scheme the origin shifting is only needed twice for all roots, compared with many more in the Lehmer–Schur algorithm. © 1993 Academic Press, Inc.

INTRODUCTION

Lehmer [4] introduced a direct global method for computing the roots of polynomials. His method uses a search procedure which is based on a theorem of Schur [10] and its extension by Cohn [2]. The Lehmer–Schur method requires the covering of the unit circle by six disks. The disk containing a root of the polynomial is determined by moving the origin of the polynomial to the center of this disk. Repeating this covering procedure k times it is possible to locate a root inside a disk of radius $(\frac{5}{12})^k$. Improvements on the Lehmer–Schur technique have also been reported by Stewart [11]. As noted by Schmidt and Rabiner [9], extracting roots of high degree polynomials is a very difficult task. They compared standard root finding methods like Jenkins and Traub [3] for the case of real roots only. Lindsey [6] studied the roots of a class of high degree band-limited polynomials, which arise in signal processing applications.

We suggest here a different approach, using the same basic ideas of the Schur–Cohn algorithm. However, rather than moving the origin of the polynomial we isolate a certain root to lie within a thin concentric annulus, obtained through a recursive bisection scheme of the radii. By perturbing the origin of a real polynomial along the real axis, another concentric annulus is obtained relative to this new origin. The intersection of the two annuli represents the location of the desired root. The accuracy of this root locating scheme is dependent on how well conditioned the roots are for the given polynomial.

Although the main ideas are described here, a related paper [7] describes a spectral factorization method for determining the roots of a polynomial.

The algorithm described in this paper, as for the Lehmer–Schur method, is globally convergent. Our scheme is, however, faster, since for a prescribed accuracy the number of Schur–Cohn transformations in the Lehmer–Schur procedure is at least six times (and possibly many more times) that used in the present scheme. It should also be noted that changing the origin of high degree polynomials may cause undesirable disturbances in some roots, particularly if the change is large enough. This situation is more likely to occur in the Lehmer–Schur scheme than in the present one.

Finally it might also be pointed out that both schemes are robust methods for root determination of high order polynomials, the accuracy of which is dependent on how well-conditioned the roots are. Poorly conditioned roots such as clusters or multiple roots may lead to less accurate estimates, as illustrated in the numerical examples. Unlike iterative root finding procedures the algorithm presented here clearly indicates the condition of the roots.

THE SCHUR–COHN TRANSFORMATION

The basic tool used in the present root detection method consists of a series of non-linear transformations in the polynomial coefficients. Each such transformation reduces the order of an Nth-degree polynomial $P_N(z)$ to an $(N-1)$th-degree polynomial. The process is then continued recursively to reduce the degree of the transformed polynomials to $N-2, N-3, ..., 2, 1$.

0021-9991/93 $5.00

Let $A_N(z) = P_N(z)$ denote the original polynomial of degree N scaled so that the constant term or lowest coefficient $a_0^{(N)} = 1$. Then the sequence of Schur–Cohn transformations is given by

$$A_{j-1}(z) = \frac{1}{1-|r_j|^2}\,[A_j(z) - r_j A_j^*(z)], \qquad j = N, N-1, ..., 2, \tag{1}$$

where the $A_j(z)$ are complex polynomials of degree j in the complex variable z defined as

$$A_j(z) = a_0^{(j)} + a_1^{(j)}z + a_2^{(j)}z^2 + \cdots + a_j^{(j)}z^j \tag{2}$$

and $A_j^*(z) = z^j \bar{A}_j(1/z)$ is its conjugate reverse defined by

$$A_j^*(z) = \bar{a}_j^{(j)} + \bar{a}_{j-1}^{(j)}z + \cdots + \bar{a}_j^{(j)}z^{j-1} + \bar{a}_0^{(j)}z^j. \tag{3}$$

The bar over a variable denotes the complex conjugate. The highest and lowest coefficients of each $A_j(z)$ are set equal to

$$a_j^{(j)} = r_j, \qquad a_0^{(j)} = 1. \tag{4}$$

It is easily seen from Eqs. (1) to (3) that the constraints applied in (4) are fulfilled for any $j \leqslant N$.

This set of transformations was introduced by Schur [10] and Cohn [2]. The r_j are known to be the reflection coefficients for acoustic wave propagation in a plane layered medium (e.g., Claerbout [1]) and so it is convenient to refer to them in this paper as reflection coefficients.

Replacing z by z^{-1} in Eq. (1) and solving these two sets of equations lead to

$$A_j(z) = A_{j-1}(z) + r_j z A_{j-1}^*(z), \qquad j = 2, ..., N; \qquad A_0(z) = 1. \tag{5}$$

The above equation is known as the Levinson recursion and is used to solve Toeplitz systems of equations in a very efficient manner [5]. The Toeplitz matrix $\{R\}_{i,j} = R_{|i-j|}$ contains the auto correlation of the seismogram series of increasing lags $|i-j| \leqslant N$.

A geometric proof of the Levinson recursion, based on propagating rays in a plane layered acoustic model, was recently given by Loewenthal and Stoffa [8]. As was shown by Schur, in order to determine if there are any roots located inside the unit circle of the complex z-plane, we need only inspect the magnitude of the reflection coefficients r_j. It transpires that there will be no roots inside the unit circle if and only if

$$|r_j| < 1, \qquad j = 2, ..., N. \tag{6}$$

If condition (6) is not met for all $2 \leqslant j \leqslant N$ then there will be at least one root inside the unit circle.

Let us assume that for a given Nth-degree polynomial $P_N(z)$ there is at least one root inside the unit circle. If there are none we reverse conjugate the polynomial and start with $P_N^*(z)$ instead. The roots of the reverse conjugate polynomial are simply related to those of the original polynomial by

$$z_j^* = \bar{z}_j^{-1}. \tag{7}$$

The roots of $P_N^*(z)$ will now all be located inside the unit circle. The Schur–Cohn transformation will now reveal at least one reflection coefficient r_j of magnitude greater than one.

We would like to determine if there are roots located inside a circle, with the center at the origin and with some other radius, $\rho < 1$. To do this we define a new polynomial $Q_N(z)$ by

$$Q_N(z) = P_N(\rho^{-1}z) = p_0 + p_1\rho^{-1}z + p_2\rho^{-2}z^2 + \cdots + p_N\rho^{-N}z^N. \tag{8}$$

Searching for roots of $Q_N(z)$ inside the unit circle is equivalent to searching for roots of $P_N(z)$ inside the circle $|z| < \rho$.

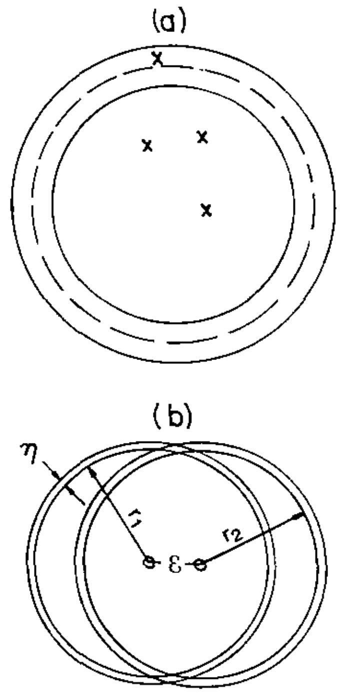

FIG. 1. (a) Inner and outer radii (solid circles) of a captured root. A bisection using the middle radius (dashed circle) is performed, revealing that the root resides between the dashed and the outer radius. (b) The magnitudes of the unperturbed (r_1) and the perturbed root (r_2) are revealed within accuracy of η. Their crossing reveals the actual root position.

The searching strategy for locating the smallest root inside the unit circle is a radius bisection method extended to two dimensions. We thus look for roots inside a circle of radius $\rho = \frac{1}{2}$ using the Schur–Cohn transformations. If there are any roots inside the circle with $\rho = \frac{1}{2}$ we continue the process with $\rho = \frac{1}{4}$. Otherwise we continue with a circle of radius $\rho = \frac{3}{4}$ and so on. After k steps we will have fixed the magnitude of the smallest root with an accuracy of at least $2^{-k} = \eta$. The radial bisection method is illustrated in Fig. 1a. Note that for reversed polynomials the inverse magnitude of the largest root will be revealed to this accuracy.

Note in the case that at the Kth step one of the reflection coefficients is, in the machine's accuracy, very close to a unit, we can try to search in the vicinity for a radius $\beta 2^{-k}$, where $\beta \sim \frac{1}{2}$. The number of bisections K cannot exceed the number of bits in the machine's mantissa. Thus, a multiple root with multiplicity m can achieve an accuracy of only $\eta = 2^{-k/m}$.

OBTAINING MAGNITUDES OF OTHER ROOTS

To find the magnitude of the rest of the roots we use the result known from Schur–Cohn theory that indicates the number of roots within the unit circle. The algorithm can be described by defining two quantities V_j and count as

$$V_{N-j} = V_{N-j+1}(1 - |r_{N-j+1}|^2), \qquad j = 1, 2, ..., N-1. \quad (9)$$

If $(V_{N-j} < 0)$ the root counter is updated by count = count + 1. The above algorithm is initialized by setting $V_N = 1$ and count = 0. Here the quantity in parentheses in Eq. (9) is the same denominator appearing in Eq. (1). The transformation indicated in Eq. (8) determines the number of roots counted for any circle centered at the origin.

Assume now that we have determined the magnitude of roots inside a circle of radius ρ_j and that the number of roots found inside this circle is count_j. In order to detect additional roots between ρ_j and the unit circle we repeat the process of two-dimensional bisection search as described previously. A new root will be found only if count > count_j.

DETERMINING THE PHASE OF THE ROOT

The phase is determined by adopting a simple intuitive approach. First the origin is slightly perturbed so that z is shifted to $z - z_0$. Then each term $(z - z_0)^j$ in the perturbed polynomial is expanded by the binomial theorem, leading to a new polynomial of the same degree. The root nearest the origin can be located from the intersection of the two annuli representing the modulus of the root in the perturbed and unperturbed polynomials.

Note that for real polynomials where complex roots appear in conjugate pairs and the origin is perturbed along the real axis, there are two points of intersection corresponding to this conjugate pair (see Fig. 1b). In the more general complex polynomial case we can obtain an approximation of the phase of the roots by an additional perturbation of the origin, say along the imaginary axis. The desired root is then located at the intersection of three annuli.

We choose to perturb the origin along the real x-axis say to $x_0 = \varepsilon$, where $\varepsilon < 1$. Let us assume that the root lies on a circle of radius r_1. For the perturbed origin the same root is assumed to lie on a circle of radius r_2. Another perturbation is carried out by an amount ε in the imaginary axis and the root now has a radius r_3. The desired root (x, y) can now be determined by solving algebraically two pairs of the equations

$$x^2 + y^2 = r_1^2; \qquad (x-\varepsilon)^2 + y^2 = r_2^2; \qquad x^2 + (y-\varepsilon)^2 = r_3^2. \quad (10)$$

The solution of Eqs. (10) is given by

$$x = \frac{r_1^2 - r_2^2}{2\varepsilon} + \frac{\varepsilon}{2}; \qquad y = \frac{r_1^2 - r_3^2}{2\varepsilon} + \frac{\varepsilon}{2}. \quad (11)$$

Assuming that the radii can be determined with an accuracy of $\eta \ll 1$, an error analysis of the equations in (11) shows that x and y have an associated error $2\eta/\varepsilon$. We will demonstrate the error analysis for the first equation in (11). We assume that r_1 and r_2 are the exact values which determine x without an error. We use, however, nonaccurate values $r_1 + \eta_1$; $r_2 + \eta_2$ which yield the value $x + \alpha$. Here $|\eta_1| \leqslant \eta$, $|\eta_2| \leqslant \eta$ and α is the error we seek. Placing the inaccurate values in (11) we find

$$x + \alpha = \frac{(r_1+\eta_1)^2 - (r_2+\eta_2)^2}{2\varepsilon} + \frac{\varepsilon}{2} \leqslant \frac{r_1^2 + 2(r_1+r_2)\,\eta - r_2^2}{2\varepsilon} + \frac{\varepsilon}{2}. \quad (12)$$

By subtracting the value of x as given in (11) and noting that $|r_1| \leqslant 1$ and $|r_2| \leqslant 1$ we find that the maximum error in determining x (and y) is given by $\alpha \leqslant 2\eta/\varepsilon$.

Note that since we have assumed that both annuli are common to the same root, ε needs to be chosen small enough to avoid the incorrect selection of another root. On the other hand, ε has to be large compared to η. If, for instance, we choose $\eta = 10^{-7}$ and assume a maximum separation of roots to be $\varepsilon = 10^{-2}$ we obtain the roots (broadly speaking) to five significant figures. This is a relative error, as can be seen from the fact that for large roots with magnitude greater than units we determine only their inverse conjugates.

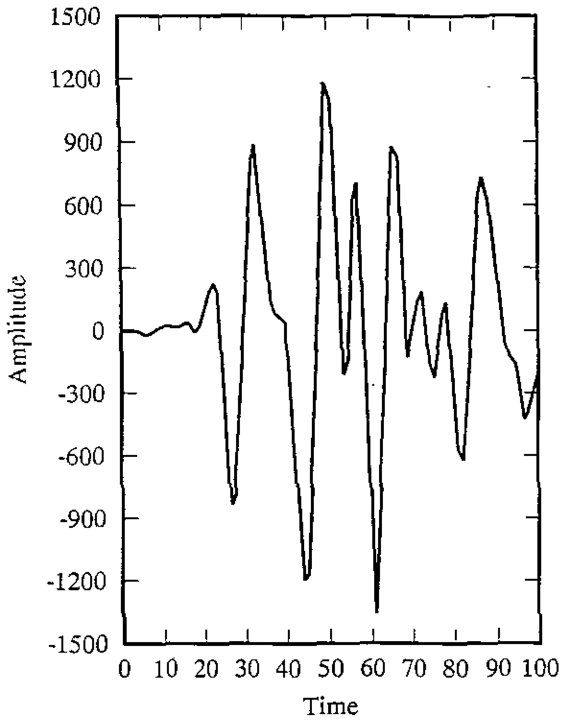

FIG. 2. The time sampled band-limited seismogram.

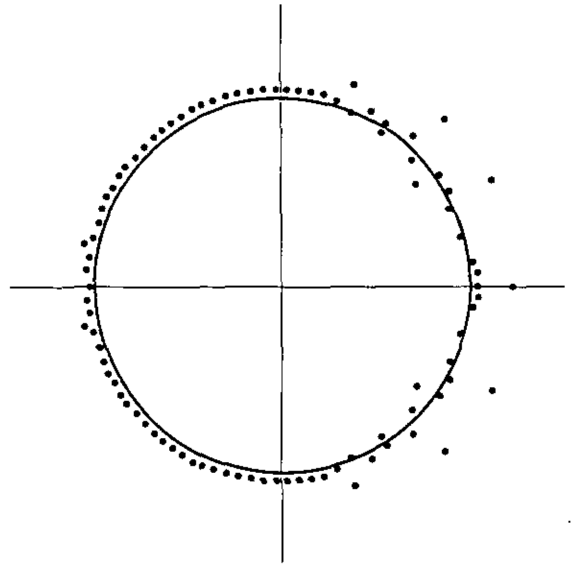

FIG. 3. The complex roots associated with the seismogram of Fig. 1.

NUMERICAL EXAMPLES

In order to illustrate the proposed scheme we have chosen three examples. The first two are ninth-degree real polynomials. The first of these has the prescribed complex roots

$$0.2 + i0.0; \quad 0.21 + i0.0; \quad 0.3 \pm i0.1;$$
$$0.9 \pm i0.6; \quad 0.0 \pm i0.7; \quad 3.0 + i0.0.$$

The second polynomial has the same set of roots, except that the root $0.21 + i0.0$ is changed to $0.2 + i0.0$, making this a double root.

The roots were found to a rough approximation with the method presented in this paper. They were then iterated by the Newton–Raphson method to obtain a more accurate approximation. The constants η and ε referred to in the text were set to be $\eta = 0.0005$ and $\varepsilon = 0.01$, respectively. The computations for the first two examples were performed on a 32-bit single precision computer. The third example was computed on a 60-bit machine.

Table I, which lists the results of the computations, contains only a selection of the roots, in fact, the four roots with the smallest magnitude. The other roots do not add anything of significance to the present study. Note that in the second example (presented in the lower part of Table I) we could not bisect more than eight times at the double root, since one of the reflection coefficients becomes too close to unit magnitude. The effective η has therefore decreased. This is due to the fact that a double root can only be determined numerically with half the number of significant digits of an isolated well-behaved root.

As our third example we have chosen a 98th-degree poly-

TABLE I

First Four Roots of Example Polynomials 1 and 2

	No. bisections		Root magnitude		Root estimates	
Exact roots	Unpert	Pert	Unpert	Pert	Rough	Iterated
$0.2 + i0$	14	13	0.20001	0.19006	$0.19958 + i.00125$	$0.200002 - i.00000008$
$0.21 + i0$	14	14	0.21002	0.20001	$0.21018 + i.00031$	$0.209999 - i.00000002$
$0.3 \pm i.1$	13	13	0.31628	0.30676	$0.30155 \pm i.096$	$0.3000003 \pm i.099999$
$0.2 + i0^{(2)}$	7	8	0.1992	0.1894	$0.1991 + i.0003$	$0.2002 + i.00006$
$0.3 + i.1$	13	13	0.3160	0.3065	$0.3013 + i.0961$	$0.2999999 + i.1000003$

Note. Example polynomial 1 is shown in the upper part of the table; polynomial 2 is in the lower part. Also included are the results for the unperturbed and perturbed computations.

nomial constructed from 98 consecutive samples of a real seismogram. The jth time sample of the seismogram is the jth coefficient of the polynomial (see Fig. 2). We are interested in determining and investigating the roots of this polynomial. A similar example constructed from a time series was performed by Schmidt and Rabiner [9]. For this example we have chosen $\varepsilon = 0.01$ and have selected up to 20 bisection steps to determine the magnitude of the roots. This gives an accuracy of the root location to five significant figures.

The 98 roots are sketched relative to the unit circle in Fig. 3. The reason so many roots are located along the unit circle is due to the finite bandwidth of the underlying seismogram.

CONCLUSION

A new robust method for determining the roots of a polynomial has been presented. Like the Lehmer–Schur method, it is based on the Schur–Cohn transformation. However, the current method significantly reduces the number of such transformations needed to locate the roots within a prescribed accuracy. Only single (double) origin perturbations are needed for real (complex) polynomials. The origin perturbations are kept small, ensuring more stable root approximations.

The merit of the new algorithm presented here is that it gives an approximation to the roots of high order polynomials. Whereas algorithms such as the Newton–Raphson and Jenkins–Traub methods can yield very accurate approximations, they usually fail in finding roots of very high degree polynomials. In a sense, the method given here trades off high accuracy for the high degree of the polynomials considered.

ACKNOWLEDGMENT

The author thanks Dr. Peter Buchen (University of Sydney, Australia) for critically reading and amending the manuscript.

REFERENCES

1. J. F. Claerbout, *Fundamentals of Geophysical Data Processing* (McGraw–Hill, New York, 1976).
2. A. Cohn, *Math. Z.* **14**, 110 (1922).
3. M. A. Jenkins and J. F. Traub, *SIAM J. Numer. Anal.* **7**, 545 (1970).
4. D. H. Lehmer, *J. Assoc. Comput. Mach.* **8**, 151 (1961).
5. N. Levinson, *J. Math. Phys.* **25**, 261 (1947).
6. J. P. Lindsey, *Geophys. The Leading Edge of Exploration.* July, 10 (1988).
7. D. Loewenthall, "Numerical Computation of the Roots of Polynomials by Spectral Factorization," in *Topics in Numerical Analysis III*, edited by J. J. H. Miller (Academic Press, New York, 1977).
8. D. Loewenthal and P. L. Stoffa, *J. Acoust. Soc. Am.* **90**, 1101 (1991).
9. C. E. Schmidt and L. R. Rabiner, *IEEE Trans. Acoust. Speech Signal Process.* Feb., 96 (1977).
10. I. Schur, *J. Reine Angew. Math.* **147**, 205, 122 (1917).
11. G. W. Stewart, III, *Math. Comput.* **25**, 829 (1969).